

ASYMPTOTIC EXPANSION OF THE WAVELET TRANSFORM FOR SMALL a

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Abstract

Asymptotic expansion of the wavelet transform for small values of the dilation parameter a is obtained using asymptotic expansion of the Mellin convolution technique of Wong. Asymptotic expansions of Morlet wavelet transform, Mexican hat wavelet transform and Haar wavelet transform are obtained as special cases.

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1 Introduction

The wavelet transform of f with respect to the wavelet ψ is defined by

$$(W_\psi f)(b, a) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad b \in \mathbb{R}, a > 0, \quad (1)$$

provided the integral exists [1].

For fixed parameter $b \in \mathbb{R}$ in (1) the family $\{\psi(\frac{\bullet-b}{a}) : a > 0\}$ zooms in an every detail of in a neighborhood of b as long as a is sufficiently small. The frequency resolution is controlled by the parameter a and for small a , $(W_\psi f)(b, a)$ represents the frequency components of the signal f . Therefore, it is highly desirable to know the asymptotic behavior of $(W_\psi f)(b, a)$ for small values of a . Using Fourier transform (1) can also be expressed as

$$(W_\psi f)(b, a) = \frac{\sqrt{a}}{2\pi} \int_{-\infty}^{\infty} e^{ib\omega} \hat{f}(\omega) \overline{\hat{\psi}(a\omega)} d\omega, \quad (2)$$

where

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t) dt.$$

Putting $a = \frac{1}{c}$, from (2) we have

$$(W_\psi f)(b, a) = \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} e^{ibcu} \hat{f}(cu) \overline{\hat{\psi}(u)} du, \quad (3)$$

Asymptotic expansion with explicit error term for the general integral

$$I(x) = \int_0^{\infty} g(t) h(xt) dt, \quad (4)$$

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as $x \rightarrow \infty+$, was obtained by Wong [5], [6] under different conditions on g and h . The asymptotic expansion for (3) can be obtained by setting $h(t) = e^{ibt} \hat{f}(t)$ for fixed $b \in \mathbb{R}$. Let us recall basic result from [6] that will be used in the present investigation.

Here we assume that $g(t)$ has an expansion of the form

$$g(t) \sim \sum_{s=0}^{n-1} c_s t^{s+\lambda-1}, \text{ as } t \rightarrow 0+, \quad (5)$$

where $0 < \lambda \leq 1$. Regarding the function h , we assume that as $t \rightarrow 0+$,

$$h(t) = O(t^\rho), \quad \rho + \lambda > 0, \quad (6)$$

and that as $t \rightarrow +\infty$

$$h(t) \sim \exp(i\tau t^p) \sum_{s=0}^{\infty} b_s t^{-s-\beta}, \quad (7)$$

where $\tau \neq 0$ is real, $p \geq 1$ and $\beta > 0$. Let $M[h; z]$ denote the generalized Mellin transform of h defined by

$$M[h; z] = \lim_{\epsilon \rightarrow 0+} \int_0^\infty t^{z-1} h(t) \exp(-\epsilon t^p) dt. \quad (8)$$

This together with (4) and [6, p.217], gives

$$I(x) = \sum_{s=0}^{n-1} c_s M[h; s + \lambda] x^{-s-\lambda} + \delta_n(x), \quad (9)$$

where

$$\delta_n(x) = \lim_{\epsilon \rightarrow 0+} \int_0^\infty g_n(t) h(xt) \exp(-\epsilon t^p) dt. \quad (10)$$

If we now define recursively $h^\circ(t) = h(t)$ and

$$h^{(-j)}(t) = - \int_t^\infty h^{(-j+1)}(u) du, \quad j = 1, 2, \dots,$$

then conditions of validity of aforesaid results are given by the following [6, Theorem 6, p.217].

Theorem 1. Assume that (i) $g^{(m)}(t)$ is continuous on $(0, \infty)$, where m is a non-negative integer; (ii) $g(t)$ has an expansion of the form (5), and the expansion is m times differentiable; (iii) $h(t)$ satisfies (6) and (7) and (iv) and as $t \rightarrow \infty$, $t^{-\beta} g^{(j)}(t) = O(t^{-1-\epsilon})$ for $j = 0, 1, \dots, m$ and for some $\epsilon > 0$. Under these conditions, the result (9) holds with

$$\delta_n(x) = \frac{(-1)^m}{x^m} \int_0^\infty g_n^{(m)}(t) h^{(-m)}(xt) dt, \quad (11)$$

where n is the smallest positive integer such that $\lambda + n > m$.

The asymptotic expansion for the wavelet transform (2) for large values of dilation parameter a has already been obtained in [3].

The aim of the present paper is to derive asymptotic expansion of the wavelet transform given by (2) for small values of a , using formula (9).

2 ASYMPTOTIC EXPANSION FOR SMALL a

In this section using aforesaid technique, we obtain asymptotic expansion of $(W_\psi f)(b, a)$ for small values of a , keeping b fixed. We have

$$\begin{aligned}
 (W_\psi f)(b, a) &= \frac{\sqrt{c}}{2\pi} \int_{-\infty}^{\infty} e^{ibcu} \overline{\hat{\psi}}(u) \hat{f}(cu) du \\
 &= \frac{\sqrt{c}}{2\pi} \left\{ \int_0^{\infty} e^{ibcu} \overline{\hat{\psi}}(u) \hat{f}(cu) du \right. \\
 &\quad \left. + \int_0^{\infty} e^{-ibcu} \overline{\hat{\psi}}(-u) \hat{f}(-cu) du \right\} \\
 &= \frac{\sqrt{c}}{2\pi} (I_1 + I_2), \quad (say).
 \end{aligned} \tag{12}$$

Let us set

$$h(u) = e^{ibu} \hat{f}(u). \tag{13}$$

Assume that

$$\hat{f}(u) \sim \sum_{r=0}^{\infty} b_r u^{-r-\beta}, \quad \beta > 0, \quad u \rightarrow \infty;$$

so that

$$h(u) \sim e^{ibu} \sum_{r=0}^{\infty} b_r u^{-r-\beta}, \quad \beta > 0, \quad u \rightarrow \infty, b \neq 0. \tag{14}$$

$$\overline{\hat{\psi}}(u) \sim \sum_{s=0}^{\infty} c_s u^{s+\lambda-1}, \quad \text{as } u \rightarrow 0.$$

For $n \geq 1$, we write

$$\overline{\hat{\psi}}(u) \sim \sum_{s=0}^{n-1} c_s u^{s+\lambda-1} + \overline{\hat{\psi}}_n(u), \tag{15}$$

where $0 < \lambda \leq 1$. Also assume that

$$h(u) = O(u^\rho), \quad u \rightarrow 0, \rho + \lambda > 0. \tag{16}$$

The generalized Mellin transform of h is defined by

$$M[h; z_1] = \lim_{\varepsilon \rightarrow 0+} \int_0^{\infty} u^{z_1-1} h(u) e^{-\varepsilon u} du. \tag{17}$$

Then by (9),

$$I_1(c) = \sum_{s=0}^{n-1} c_s M[h; s + \lambda] c^{-s-\lambda} + \delta_n^1(c), \tag{18}$$

where

$$\delta_n^1(c) = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty \overline{\hat{\psi}_n}(u) h(cu) e^{-\varepsilon u} du, \quad (19)$$

and, from (17)

$$M[h(-u); z_1] = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty u^{z_1-1} h(-u) e^{-\varepsilon u} du.$$

Hence

$$I_2(c) = \sum_{s=0}^{n-1} c_s (-1)^{s+\lambda+1} M[h(-u); s+\lambda] c^{-s-\lambda} + \delta_n^2(c), \quad (20)$$

where

$$\delta_n^2(c) = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty \overline{\hat{\psi}_n}(-u) h(-cu) e^{-\varepsilon u} du. \quad (21)$$

Finally, from (12), (18) and (20) we get the asymptotic expansion:

$$\begin{aligned} (W_\psi f)(b, a) &= \frac{\sqrt{c}}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s \left(M[h(u); s+\lambda] + (-1)^{s+\lambda+1} M[h(-u); s+\lambda] \right) \right. \\ &\quad \left. \times c^{-s-\lambda} + \delta_n^1(c) + \delta_n^2(c) \right\} \end{aligned}$$

Finally, setting $c = \frac{1}{a}$ we get the asymptotic expansion for small values of a :

$$\begin{aligned} (W_\psi f)(b, a) &= \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s \left(M[h(u); s+\lambda] + (-1)^{s+\lambda+1} M[h(-u); s+\lambda] \right) \right. \\ &\quad \left. \times a^{s+\lambda-1/2} + \delta_n(a) \right\}, \quad (22) \end{aligned}$$

where

$$\begin{aligned} \delta_n(a) &= \frac{1}{\sqrt{a}} \lim_{\varepsilon \rightarrow 0+} \left(\int_0^\infty \overline{\hat{\psi}_n}(u) h(u/a) e^{-\varepsilon u} du \right. \\ &\quad \left. + \int_0^\infty \overline{\hat{\psi}_n}(-u) h(-u/a) e^{-\varepsilon u} du \right). \quad (23) \end{aligned}$$

Using Theorem 1. we get the following existence theorem for formula (22).

Theorem 2. Assume that (i) $\overline{\hat{\psi}}^{(m)}(u)$ is continuous on $(-\infty, \infty)$, where m is a nonnegative integer; (ii) $\overline{\hat{\psi}}(u)$ has asymptotic expansion of the form (15) and the expansion is m times differential; (iii) $h(u)$ satisfies (14) and (16) and (iv) as $u \rightarrow \infty$ $u^{-\beta} \overline{\hat{\psi}}^{(m)}(u) = O(u^{-1-\epsilon})$ for $j = 0, 1, 2, \dots, m$ and for some $\epsilon > 0$. Under these conditions, the result (22) holds with

$$\delta_n(a) = (-1)^m a^{m-1/2} \int_{-\infty}^\infty \overline{\hat{\psi}_n}^{(m)}(u) h^{(-m)}(u/a) e^{-\varepsilon u} du.$$

where n is the smallest positive integer such that $\lambda + n > m$.

In the following sections we shall obtain asymptotic expansions for certain special cases of the general wavelet transform.

3 MORLET WAVELET TRANSFORM

In this section we shall exploit the following result [4, eq.(12) p.57] for series manipulation

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k, n) t^{n+2k} = \sum_{n=0}^{\infty} \sum_{k=0}^{[n/2]} A(k, n-2k) t^{n-2k}. \quad (24)$$

We choose

$$\psi(t) = \sqrt{2\pi} e^{iu_0 t - t^2/2}. \quad (25)$$

Then from [1, p.373]

$$\hat{\psi}(u) = \sqrt{2\pi} e^{\frac{-(u-u_0)^2}{2}}. \quad (26)$$

Now, using (24) we can write $\hat{\psi}(u)$ in form of (15)

$$\begin{aligned} \hat{\psi}(u) &= \sqrt{2\pi} e^{-u_0^2/2} e^{u_0 u} e^{-u^2/2} \\ &= \sqrt{2\pi} e^{-u_0^2/2} \sum_{s=0}^{\infty} \frac{(u_0 u)^s}{s!} \sum_{p=0}^{\infty} \frac{(-1)^p u^{2p}}{2^p p!} \\ &= \sqrt{2\pi} e^{-u_0^2/2} \sum_{s=0}^{\infty} \sum_{p=0}^{\infty} \frac{(-1)^p u_0^s u^{s+2p}}{s! 2^p p!} \\ &= \sqrt{2\pi} e^{-u_0^2/2} \sum_{s=0}^{\infty} \sum_{p=0}^{[s/2]} \frac{(-1)^p u_0^{s-2p} u^s}{p! (s-2p)! 2^p} \\ &= \sum_{s=0}^{\infty} c_s u^s + \hat{\psi}(u), \end{aligned} \quad (27)$$

where

$$c_s = \sqrt{2\pi} e^{-u_0^2/2} \sum_{p=0}^{[s/2]} \frac{(-1)^p u_0^{s-2p}}{p! (s-2p)! 2^p}. \quad (28)$$

Thus $\hat{\psi}(u)$ possesses asymptotic expansion (15) with $\lambda = 1$ and c_s given by (26). Hence, using (22) and (26) we get the following asymptotic expansion of $(W_\psi f)(b, a)$ for small values of a .

$$\begin{aligned} (W_\psi f)(b, a) &= \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s (M[h(u) : s+1] + (-1)^s M[h(-u) : s+1]) \right. \\ &\quad \left. \times a^{s+1/2} + \delta_n(a) \right\}, \end{aligned} \quad (29)$$

where

$$\begin{aligned}\delta_n(a) &= \frac{1}{\sqrt{a}} \lim_{\varepsilon \rightarrow 0^+} \left(\int_0^\infty \hat{\psi}_n(u) h(u/a) e^{-\varepsilon u} du \right. \\ &\quad \left. + \int_0^\infty \hat{\psi}_n(-u) h(-u/a) e^{-\varepsilon u} du \right).\end{aligned}\quad (30)$$

Using Theorem 2. we get the following existence theorem for formula (29).

Theorem 3. *Assume that $h(u)$ satisfies (14) and (16). Under these conditions, the result (29) holds with*

$$\delta_n(a) = (-1)^m a^{m-1/2} \int_{-\infty}^\infty \hat{\psi}_n^{(m)}(u) h^{(-m)}(u/a) e^{-\varepsilon u} du.$$

where n is the smallest positive integer such that $1 + n > m$.

4 MEXICAN HAT WAVELET TRANSFORM

In this section we choose

$$\psi(t) = (1 - t^2) e^{-t^2/2}.$$

Then from [1, p. 372]

$$\hat{\psi}(u) = \sqrt{2\pi} u^2 e^{-u^2/2}.$$

Now,

$$\begin{aligned}\hat{\psi}(u) &= \sqrt{2\pi} u^2 \sum_{r=0}^\infty \frac{(-1)^r u^{2r}}{2^r r!} \\ &= \sqrt{2\pi} \sum_{l=1}^\infty \frac{(-1)^{l-1} u^{2l}}{2^{l-1} (l-1)!} \\ &= \sum_{s=0}^\infty c_s u^s,\end{aligned}\quad (31)$$

where

$$c_s = \begin{cases} \sqrt{2\pi} \frac{(-1)^{l-1}}{2^{l-1} (l-1)!} & \text{if } s = 2l, l = 1, 2, 3, \dots \\ 0 & \text{otherwise.} \end{cases}\quad (32)$$

Then, from (22) with $\lambda = 1$ and c_s given by (32) yields the asymptotic expansion:

$$\begin{aligned}(W_\psi f)(b, a) &= \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s (M[h(u) : s+1] + (-1)^s M[h(-u) : s+1]) \right. \\ &\quad \left. \times a^{s+1/2} + \delta_n(a) \right\},\end{aligned}\quad (33)$$

where

$$\begin{aligned}\delta_n(a) &= \frac{1}{\sqrt{a}} \lim_{\varepsilon \rightarrow 0^+} \left(\int_0^\infty \hat{\psi}_n(u) h(u/a) e^{-\varepsilon u} du \right. \\ &\quad \left. + \int_0^\infty \hat{\psi}_n(-u) h(-u/a) e^{-\varepsilon u} du \right).\end{aligned}\tag{34}$$

Using Theorem 2. we get the following existence theorem for formula (33).

Theorem 4. *Assume that $h(u)$ satisfies (2.3) and (2.5). Under these conditions, the result (32) holds with*

$$\delta_n(a) = (-1)^m a^{m-1/2} \int_{-\infty}^\infty \hat{\psi}_n^{(m)}(u) h^{(-m)}(u/a) e^{-\varepsilon u} du,\tag{35}$$

where n is the smallest positive integer such that $1 + n > m$.

5 HAAR WAVELET TRANSFORM

let us choose

$$\psi(t) = \begin{cases} 1, & 0 \leq t < 1/2 \\ -1, & 1/2 \leq t < 1 \\ 0, & \text{otherwise,} \end{cases}$$

Then from [1, p. 368],

$$\begin{aligned}\overline{\hat{\psi}}(au) &= 4ie^{-iu/2} \frac{\sin^2 u/4}{u} \\ &= \frac{i}{u} (1 - 2e^{iu/2} + e^{iu}) \\ &= \frac{i}{u} \left(1 - 2 \sum_{r=0}^\infty \frac{(iu)^r}{2^r r!} + \sum_{r=0}^\infty \frac{(iu)^r}{r!} \right) \\ &= \sum_{r=1}^\infty \frac{i^{r+1} u^{r-1}}{r!} \left(1 - \frac{1}{2^{r-1}} \right) \\ &= \sum_{s=0}^\infty \frac{i^{s+2} u^s}{(s+1)!} \left(1 - \frac{1}{2^s} \right) \\ &= \sum_{s=0}^\infty c_s u^s,\end{aligned}\tag{36}$$

where

$$c_s = \frac{i^{s+2}}{(s+1)!} \left(1 - \frac{1}{2^s} \right).\tag{37}$$

Then, from (22) with $\lambda = 1$ and c_s given by (37) we get

$$(W_\psi f)(b, a) = \frac{1}{2\pi} \left\{ \sum_{s=0}^{n-1} c_s (M[h(u); s+1] + (-1)^s M[h(-u); s+1]) \right. \\ \left. \times a^{s+1/2} + \delta_n(a) \right\}, \quad (38)$$

where

$$\delta_n(a) = \frac{1}{\sqrt{a}} \lim_{\varepsilon \rightarrow 0+} \left(\int_0^\infty \overline{\hat{\psi}_n}(u) h(u/a) e^{-\varepsilon u} du \right. \\ \left. + \int_0^\infty \overline{\hat{\psi}_n}(-u) h(-u/a) e^{-\varepsilon u} du \right). \quad (39)$$

6 ASYMPTOTIC EXPANSION FOR SMALL a CONTINUED

In this section we obtain asymptotic expansion of the wavelet transform given in the form (1) when $a \rightarrow 0+$. Naturally, in this case we have to impose conditions on f and ψ instead of \hat{f} and $\hat{\psi}$.

Now, let us write (1) in the form:

$$(W_\psi f)(b, a) = c^{1/2} \int_{-\infty}^\infty f(t+b) \overline{\psi(ct)} dt, \quad (40)$$

where $c = 1/a \rightarrow +\infty$ and b is assumed to be a fixed real number. Then setting $g(t) = f(t+b)$ and $h(t) = \overline{\psi(t)}$, we have

$$(W_\psi f)(b, a) = c^{1/2} \left[\int_0^\infty g(t) h(ct) dt + \int_{-\infty}^0 g(t) h(ct) dt \right] \\ = c^{1/2} [I_1 + I_2] \quad (\text{say}). \quad (41)$$

Assume that $g(t)$ satisfies (5) and $h(t)$ satisfies (6) and (7). Then from (9) it follows that

$$I_1 = \sum_{s=0}^{n-1} c_s M[\bar{\psi}; s+\lambda] c^{-s-\lambda} + \delta_n^1(a), \quad (42)$$

where

$$\delta_n^1(a) = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty g_n(t) \overline{\psi(t/a)} e^{-\varepsilon t} dt; \quad (43)$$

and

$$I_2 = \sum_{s=0}^{n-1} c_s (-1)^{s+\lambda-1} M[\overline{\psi(-t)}; s+\lambda] + \delta_n^2(a), \quad (44)$$

where

$$\delta_n^2(a) = \lim_{\varepsilon \rightarrow 0+} \int_0^\infty g_n(t) \overline{\psi(-t/a)} e^{-\varepsilon t^p} dt. \quad (45)$$

From (41), (42) and (44) we get

$$\begin{aligned} (W_\psi f)(b, a) &= \sum_{s=0}^{n-1} c_s M[\bar{\psi}; s + \lambda] a^{s+\lambda-1/2} \\ &\quad + \sum_{s=0}^{n-1} c_s (-1)^{s+\lambda-1} M[\overline{i\psi(-t)}; s + \lambda] \\ &\quad \times a^{s+\lambda-1/2} + \delta_n(a), \end{aligned} \quad (46)$$

where

$$\begin{aligned} \delta_n(a) &= a^{-1/2} \lim_{\varepsilon \rightarrow 0+} \left\{ \int_0^\infty g_n(t) \overline{\psi(t/a)} e^{-\varepsilon t^p} dt \right. \\ &\quad \left. + \int_0^\infty g_n(-t) \overline{\psi(-t/a)} e^{-\varepsilon t^p} dt \right\}. \end{aligned} \quad (47)$$

7 Example

Let us find again asymptotic expansion of Morlet wavelet transform for small a , using the above technique. Here

$$\psi(t) = e^{i\omega_0 t - t^2/2}.$$

Suppose that $g(t) = f(t + b)$ satisfies (5). Then from (42), using formula [2, eq.(21), p.16], we get

$$\begin{aligned} I_1 &= \sum_{s=0}^{n-1} c_s M[\bar{\psi}; s + \lambda] c^{-s-\lambda} + \delta_n^1(a) \\ &= \sum_{s=0}^{n-1} c_s \int_0^\infty e^{i\omega_0 t - t^2/2} t^{s+\lambda-1} dt a^{s+\lambda} + \delta_n^1(a) \\ &= \sum_{s=0}^{n-1} c_s \Gamma(s + \lambda) e^{-\omega_0^2/4} D_{-s-\lambda}(-i\omega_0) a^{s+\lambda} + \delta_n^1(a). \end{aligned} \quad (48)$$

where $D_\nu(z)$ denoted parabolic cylinder function. Similarly,

$$I_2 = \sum_{s=0}^{n-1} c_s (-1)^{s+\lambda-1} \Gamma(s + \lambda) e^{-\omega_0^2/4} D_{-s-\lambda}(i\omega_0) a^{s+\lambda} + \delta_n^2(a). \quad (49)$$

Therefore,

$$(W_\psi)(b, a) = \sum_{s=0}^{n-1} c_s \Gamma(s + \lambda) e^{-\omega_0^2/4} (D_{-s-\lambda}(-i\omega_0) + (-1)^{s+\lambda-1} D_{-s-\lambda}(i\omega_0)) \times a^{s+\lambda-1/2} + \delta_n(a), \quad (50)$$

where

$$\delta_n(a) = a^{-1/2} \left\{ \int_0^\infty g_n(t) e^{-i\omega_0(t/a) - (t/a)^2/2} dt + \int_0^\infty g_n(-t) e^{i\omega_0(t/a) - (t/a)^2/2} dt \right\}. \quad (51)$$

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